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On Everywhere Convergence of Sequences of Convolution Kernels

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INTRODUCTION

Let F be a differentiable function of a real variable and let $f = F'$. According to standard theorems, f is in the first class of Baire (i.e., f is the limit of a sequence of continuous functions) and f has the Darboux (i.e., the intermediate value) property. That f is in the first class of Baire is readily seen by observing that for each x,

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \text{where} \quad f_n(x) = \frac{F\left(x + \frac{1}{n}\right) - F(x)}{\frac{1}{n}}$$

The usual proof that f has the Darboux property involves the observation that if $f(x_1) < 0 < f(x_2)$, then F attains a maximum or a minimum at some point x_3 between x_1 and x_2 and, therefore, $f(x_3) = 0$.

An entirely different proof of these results is possible if one assumes that $f \in L_1$. This proof involves convolution kernels and allows a generalization which asserts that if a function $f \in L_1$ can be everywhere summed in a certain sense by convolution kernels, then f is in the first class of Baire and has the Darboux property. The purpose of this article is to motivate this generalization (proved in Section 3) and to apply it to Poisson kernels in order to prove a theorem concerning the boundary behavior of certain harmonic functions (Section 4).

2. MOTIVATION

Let $f \in L_1[a, b]$, and define a function F by

$$F(x) = \int_a^x f(t)\, dt.$$

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The symmetric derivative of f is defined by

$$F_s'(x) = \lim_{h\to 0} \frac{1}{2h} [F(x+h) - F(x-h)] = \lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt$$
$$= \lim_{h\to 0} \int_a^b f(t)\, K_h(x-t)\, dt$$

provided this limit exists. Here $K_h(t) = (2h)^{-1} \chi_{[-h,h]}(t)$, where $\chi_{[-h,h]}$ is the characteristic function of the interval $[-h, h]$.

Now, it is possible for $F_s'(x) = f(x)$ to hold everywhere without f having the Darboux property. For example, if we define f by

$$f(x) = \begin{cases} 0 & \text{for} \quad -\pi \leqslant x < 0 \\ 1 & \text{for} \quad x = 0 \\ 2 & \text{for} \quad 0 < x < \pi, \end{cases}$$

then $F_s'(x) = f(x)$ for all $x \in [-\pi, \pi)$, but f does not have the Darboux property.

Thus the symmetric differentiation kernels K_h can sum a function everywhere without the function having the Darboux property. The same is true for the Direchlet and Fejér kernels (the function f above is everywhere the sum of its Fourier series). Of course, f is not everywhere the derivative of F, since F' does not exist for $x = 0$. By properly removing the bias to the center of an interval presented by the symmetric derivative, one can obtain the ordinary derivative in terms of the symmetric differentiation kernels. Thus, for each h define f_h by

$$f_h(x) = \int_{-\pi}^{\pi} f(t)\, K_h(x-t)\, dt.$$

If $\lim f_h(x_h) = f(x)$, whenever $x_h \to x$ in such a way that $|x - x_h| \leqslant h$, then $F'(x) = f(x)$. In effect, each point x "borrows" some of the kernels centered at nearby points in such a way as to remove the bias.

The idea of "borrowing" kernels in this manner might seem artificial at first glance, but it arises in natural ways (Section 4).

3. Main Result

For convenience, we shall deal with functions defined on the interval $[-\pi, \pi)$ and extended them periodically with period 2π to the real line R.

For our purposes, it is sufficient to deal with sequences of kernels (rather than generalized sequences), but our results remain valid if we modify the statement and proof of Theorem 1 in some obvious ways. Let $\{K_n\}$ be a sequence of functions satisfying the following (often assumed) conditions for convolution kernels:

(i) $K_n \in L_\infty$ on $[-\pi, \pi)$ for each n.

(ii) $K_n(t) \geqslant 0$ for all n and t.

(iii) $\int_{-\pi}^{\pi} K_n(t)\, dt = 1$ for all n.

(iv) For each interval I centered at 0, $K_n(x)$ converges uniformly to 0, as $n \to \infty$, on $[-\pi, \pi) \sim I$.

(v) $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t)\, K_n(x-t)\, dt = f(x)$ almost everywhere for each $f \in L_1$.

It follows from (iii) and (iv) that there exists a sequence $\{\gamma_N\} \downarrow 0$ such that

$$\lim_{N\to\infty} \int_{-\gamma_N}^{\gamma_N} K_N(t)\, dt = 1 \tag{1}$$

and

$$\sup\{K_N(t) : t \in [-\pi, -\gamma_N] \cup [\gamma_N, \pi)\} \to 0 \qquad \text{as} \qquad N \to \infty. \tag{2}$$

THEOREM 1. *Let $\{K_N\}$ be a sequence of functions satisfying conditions* (i)–(v) *above. Let $\{\gamma_N\} \downarrow 0$ and satisfy* (1) *and* (2). *Let $f \in L_1$ and define functions f_N by*

$$f_N(x) = \int_{-\pi}^{\pi} K_N(x-t)\, f(t)\, dt.$$

If for each x in $[-\pi, \pi)$ we have

$$\lim_{N\to\infty} f_N(x) = f(x); \quad \lim_{N\to\infty} f_N(x+\gamma_N) = f(x), \quad \text{and} \quad \lim_{N\to\infty} f_N(x-\gamma_N) = f(x),$$

then f is in the first class of Baire and has the Darboux property.

Proof. According to a standard theorem [3, p. 25], each function f_N is continuous since each $K_N \in L_\infty$. Since $f(x) = \lim_{N\to\infty} f_N(x)$ for all x, the function f is in the first class of Baire. To show that f has the Darboux property, it suffices to prove [2, p. 103] (since f is in the first class of Baire) that, for each x, if $\epsilon > 0$, $\delta > 0$, there exist x' and x'' such that $x - \delta < x' < x < x'' < x + \delta$ and $|f(x) - f(x')| \leqslant \epsilon$, $|f(x) - f(x'')| \leqslant \epsilon$. Suppose this condition fails for some $x = x_0$, ϵ, δ. Then we may assume without loss of generality that $f(x_0) = 0$, and $|f(x)| > \epsilon$ for all x in $(x_0, x_0 + \delta]$. Let $A = \{x : f(x) > \epsilon\} \cap (x_0, x_0 + \delta)$ and $B = \{x : f(x) < -\epsilon\} \cap (x_0, x_0 + \delta)$. The sets A and B are disjoint and $A \cup B = (x_0, x_0 + \delta)$. Since f is in the first class of Baire, each of these sets is of the type F_σ. It follows [4, p. 6] that either one of the sets A, B is empty, or it contains a point which is not a bilateral limit point of the set. In the former case, we have either $f(x) > \epsilon$ on $(x_0, x_0 + \delta)$, or $f(x) < -\epsilon$ on $(x_0, x_0 + \delta)$. In the latter case, there exists a point x_1 in one of the sets A, B and an interval having x_1 as an end point but contained in the other set. For simplicity, we shall assume the former case; the proof in the other case is entirely analogous. Thus, we are assuming $f(x_0) = 0$, and $f(x) > \epsilon$ for all $x \in (x_0, x_0 + \delta)$. Assume further, for simplicity of notation, that $x_0 = 0$ and that $\epsilon < \frac{1}{2}$. That being the case, let N_0 be so large that if $N \geqslant N_0$, then $2\gamma_N < \delta$,

$$\int_0^{2\gamma_N} K_N(\gamma_N - t)\, dt > 1 - \epsilon,$$

and

$$K_N(x) < \frac{\epsilon}{4 \int_{-\pi}^{\pi} |f(t)|\, dt}$$

for all $x \in [-\pi, -\gamma_N] \cup [\gamma_N, \pi)$. That this is possible follows from our hypotheses on the sequence $\{\gamma_N\}$. For $N \geqslant N_0$, we have

$$\begin{aligned} f_N(\gamma_N) &= \int_{-\pi}^{\pi} K_N(\gamma_N - t) f(t)\, dt \\ &= \int_0^{2\gamma_N} K_N(\gamma_N - t) f(t)\, dt + \int_{[-\pi, -\gamma_N] \cup [\gamma_N, \pi)} K_N(\gamma_N - t) f(t)\, dt. \end{aligned}$$

Now

$$\int_0^{2\gamma_N} K_N(\gamma_N - t) f(t)\, dt > \epsilon(1 - \epsilon) > \frac{\epsilon}{2},$$

and

$$\left| \int_{[-\pi, -\gamma_N] \cup [\gamma_N, \pi)} K_N(\gamma_N - t) f(t)\, dt \right| < \frac{\epsilon}{4}.$$

Thus $f_N(\gamma_N) > \epsilon/2 - \epsilon/4 = \epsilon/4$ for all $N \geqslant N_0$, contradicting our hypothesis $\lim_{N\to\infty} f_N(\gamma_N) = f(0) = 0$.

Remark 1. The requirements on the sequence $\{\gamma_N\}$ can be somewhat modified if we are dealing only with functions in L_∞. In that case we need only assume that $\{\gamma_N\} \downarrow 0$ and that condition (1) is met; i.e.,

$$\lim_{N\to\infty} \int_{-\gamma_N}^{\gamma_N} K_N(t)\, dt = 1.$$

Furthermore, conditions (iii) and (iv) on the kernels can be replaced by the weaker requirement that $\lim_{N\to\infty} \int_I K_N(t)\,dt = 1$ for every interval centered at 0. The proof of this result would be the same as that of Theorem 1 except that we would modify one of the estimates by observing that

$$\int_{[-\pi,-\gamma_N]\cup[\gamma_N,\pi)} K_N(\gamma_N - t)\,dt < \epsilon,$$

so that

$$\int_{[-\pi,-\gamma_N]\cup[\gamma_N,\pi]} K_N(\gamma_N - t) f(t)\,dt \leqslant \epsilon \,\|f\|_\infty .$$

Remark 2. We observe that Theorem 1 actually gives a better result than that suggested in our discussion of the symmetric derivative in Section 2. In effect, we weakened the hypothesis that $\lim f_h(x_h) = f(x)$, whenever $x_h \to x$ in such a way that $| x - x_h | \leqslant h$ to the hypothesis that such a result holds for the three special sequences $\{x\}$, $\{x + \gamma_N\}$, and $\{x - \gamma_N\}$.

4. Applications

We now apply Theorem 1 and its analog for functions in L_∞ to questions concerning the boundary behavior of harmonic functions.

Let D be the unit disk in the complex plane:

$$D = \{re^{i\theta} : 0 \leqslant r < 1, -\pi \leqslant \theta < \pi\}.$$

It is not difficult to verify that a sequence of points $\{r_n e^{i\theta_n}\}$, $\theta_n \geqslant 0$ converges tangentially to 1 if and only if $\lim_{n\to\infty} r_n = 1$, $\lim_{n\to\infty} \theta_n = 0$, and $\lim_{n\to\infty}(\theta_n/1 - r_n) = \infty$. Similarly, $\{r_n e^{i\theta_n}\}$ converges "more tangentially than any circle" to 1 if and only if $\lim_{n\to\infty} r_n = 1$, $\lim_{n\to\infty} \theta_n = 0$, and $\lim_{n\to\infty}(\theta_n^2/1 - r_n) = \infty$. Now let $P_r(\theta) = (1/2\pi)((1 - r^2)/1 + r^2 - 2r\cos\theta)$, the usual Poisson kernel; it is known to satisfy conditions analogous to (i)–(v) of Section 3. For $f \in L_1$, define F, the harmonic extension of f on D, by

$$F(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t) f(t)\,dt.$$

According to Fatou's Theorem [3], if $f \in L_1$, then

$$\lim_{r\to 1} F(re^{i\theta}) = f(\theta) \qquad \text{for almost all } \theta.$$

Theorem 1 and its analog for functions in L_∞ give information about a function f which is everywhere the limit of its Poisson extension on three special sequences. To see this, let $r_n e^{i\gamma_n}$ be a sequence of points in D such that $\lim_{n\to\infty} r_n = 1$, $\gamma_n \geqslant 0$, $\lim_{n\to\infty} \gamma_n = 0$ and $\lim_{n\to\infty}(\gamma_n/1 - r_n) = \infty$. One can verify that $\lim_{n\to\infty} \int_{-\gamma_n}^{\gamma_n} P_{r_n}(t)\, dt = 1$. If, in addition

$$\lim_{n\to\infty} \frac{\gamma_n{}^2}{1 - r_n} = \infty,$$

then $\sup\{P_{r_n}(t) : t \in [-\pi, -\gamma_n] \cup [\gamma_n, \pi)\} \to 0$ as $n \to \infty$.

Applying Theorem 1 we obtain the following result:

THEOREM 2. *Let $f \in L_1$ and let $F(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t) f(t)\, dt$. Let $\{r_n e^{i\gamma_n}\}$ be a sequence of points approaching $z = 1$ more tangentially than any circle. If, for every $\theta \in [-\pi, \pi)$,*

$$f(\theta) = \lim_{n\to\infty} F(r_n e^{i\theta}) = \lim_{n\to\infty} F(r_n e^{i(\theta-\gamma_n)}) = \lim_{n\to\infty} F(r_n e^{i(\theta+\gamma_n)}),$$

then f is in the first class of Baire and has the Darboux property.

If $f \in L_\infty$, we need only assume that $\{r_n e^{i\gamma_n}\}$ approaches $z = 1$ tangentially in order to assure the same conclusion.

One can also obtain a direct extention of the theorem that a derivative has the Darboux property. If $f \in L_1$ and F is defined by $F(x) = \int_0^x f(t)\, dt$ then f has the Darboux property provided that, for every x,

$$f(x) = \lim_{n\to\infty} n \left[F\left(x + \frac{1}{n}\right) - F(x)\right] = \lim_{n\to\infty} n \left[F(x) - F\left(x - \frac{1}{n}\right)\right]$$
$$= \lim_{n\to\infty} \frac{n}{2} \left[F\left(x + \frac{1}{n}\right) - F\left(x + \frac{1}{n}\right)\right].$$

This result follows immediately from the fact that the kernels

$$K_n(t) = (n/2)\, \chi_{[-n^{-1}, n^{-1}]}(t)$$

satisfy the conditions (i)–(v) of Section 3 and the fact that if one chooses $\gamma_N = N^{-1}$, conditions (1) and (2) are satisfied too.

5. Additional Remarks

In Section 3 we assumed that the kernels were convolution kernels. With appropriate modifications in the wording, Theorem 1 remains valid for approximate identities $K_n(x, t)$ provided that the functions

$$f_n(x) = \int_{-\pi}^{\pi} K_n(x, t) f(t)\, dt$$

are continuous. The continuity of $f_n(x)$ was needed to prove that f is in the first class of Baire. Our proof that f has the Darboux property depended on this fact because our test for the Darboux property is not valid without the assumption that f is a Baire -1 function.

It should also be possible to prove theorems analogous to Theorem 1 for functions which are defined on spaces more general than the line. There are several notions which generalize the Darboux property for such spaces [1].

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